J Glob Optim (2010) 46:423–433
DOI 10.1007/s10898-009-9443-x

Minimal infeasible constraint sets in convex integer programs

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Received: 18 February 2008 / Accepted: 8 May 2009 / Published online: 24 May 2009

Abstract In this paper we investigate certain aspects of infeasibility in convex integer programs, where the constraint functions are defined either as a composition of a convex increasing function with a convex integer valued function of n variables or the sum of similar functions. In particular we are concerned with the problem of an upper bound for the minimal cardinality of the irreducible infeasible subset of constraints defining the model. We prove that for the considered class of functions, every infeasible system of inequality constraints in the convex integer program contains an inconsistent subsystem of cardinality not greater than 2^n, this way generalizing the well known theorem of Scarf and Bell for linear systems. The latter result allows us to demonstrate that if the considered convex integer problem is bounded below, then there exists a subset of at most $2^n - 1$ constraints in the system, such that the minimum of the objective function subject to the inequalities in the reduced subsystem, equals to the minimum of the objective function over the entire system of constraints.

Keywords Feasibility · Infeasibility · Convex integer programming

1 Introduction

We consider the problem

$$\text{minimize } f_0(x) \tag{1}$$

$$\text{subject to: } x \in \mathcal{G} = \{x \in \mathbb{Z}^n \,|\, f_i(x) \le 0, i \in J = \{1, 2, \dots, m\}\} \tag{2}$$

where the functions f_i are defined as a composition of a convex nondecreasing function and a convex integer valued function. More precisely, $f_i(x) = h_i(p_i(x))$, $i \in J \cup \{0\}$, where h_i are convex increasing functions from $\mathbb{R}$ onto $(-\infty, \infty)$ and $p_i(x)$ are convex integer valued functions, or f_i are defined as a sum of finitely many functions of the latter form, although in this case all $p_i(x)$ are assumed to be bounded below over the set of remaining

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constraints and $h_i(x)$ are nondecreasing and into $\mathbb{R}$. Some of the functions in (1)–(2) may be in particular linear or convex quadratic. Literature in integer optimization [7–9,15,17,19] is mostly devoted to linear integer programming, where variety of methods have been presented and studied extensively. However, in the last several years there has been a growing interest in nonlinear (particularly convex) integer programming, which led to new theoretical and algorithmical developments, which on the other hand have produced applications of nonlinear integer programming in various areas of scientific computing, engineering, management science and operations research.

Since the problem (1)–(2) includes as a special case the linear integer programming, which is known to be NP-hard, the same remains true about the convex integer problem (1)–(2), which is generally intractable. The region defined by the relaxed system of inequalities will be denoted as $\mathcal{R}$, i.e.,

$$\mathcal{R} = \{x \in \mathbb{R}^n \,|\, f_i(x) \le 0, i \in J = \{1, 2, \ldots, m\}\}. \tag{3}$$

It is well known that while the convexity in continuous optimization problems assures that every local minimum point is a global minimum, it is not true for the integer programming problem of the form (1)–(2) even if all functions f_i, $i \in J$ are convex (e.g., linear) and objective function is convex quadratic (see, e.g., in [10]).

During formulation of the convex programming problem, particularly if it consists of a large number of constraints and variables, it is often difficult to determine whether or not the system is consistent. To the best of our knowledge, there are no known simple and efficient techniques to determine whether the model involving nonlinear constraints and/or integral constraints is correctly defined, i.e., whether the system is feasible (see [5,6,15]). Traditionally the problem of determining whether the system in (3) is consistent has been handled by methods devised to identify an initial feasible point [2,5]. They usually require the solution of some nonlinear problem which has still the same structure as the original problem, and contains one more constraint and variable.

In linear programs a common approach to analyzing infeasibility relies either on the identification of an irreducible infeasible subset of constraints (IIS), i.e., the set of constraints that is infeasible, but for which any proper subset of constraints is feasible or on identifying an infeasibility set (IN), i.e., a subset of constraints whose removal will transform the system into a feasible one. The IIS isolation methods in linear programs were studied by Chinneck and other authors in [4,5,8], and various properties of irreducible infeasible sets in quadratic and faithfully convex systems were later analyzed in [11,12].

Determining whether an integer linear program is infeasible requires the full expansion of a branch-and-bound tree with infeasibility being detected only when all of the leaf nodes prove to be infeasible.

The main objective of this paper is to investigate certain aspects of infeasibility in convex integer programs, in particular the problem of an upper bound for the minimal cardinality of the irreducible infeasible sets.

Section 2 contains some auxiliary results necessary in the proofs of theorems presented in Sect. 3, which include some results on attainability in constrained integer optimization problem obtained in [14].

Section 3 presents certain aspects of infeasibility in convex integer programs, where the constraint functions are defined either as a composition of a convex nondecreasing function h and convex integer valued function p or as a sum of the above functions. The class of convex integer valued functions includes in particular convex polynomials of n variables with integer coefficients. We prove, for the considered class of functions, that every infeasible system contains an inconsistent subsystem of at most 2^n constraints, this way generalizing the result on irreducible infeasible subsets proved by Bell in [3] and Scarf in [18], for systems of linear inequalities. The latter result allows us to demonstrate that if the problem (1)–(2) is bounded below, then there exists a subsystem of the system (2), with at most $2^n - 1$ constraints, such that the minimum of the objective function subject to the inequalities in the reduced subsystem equals to the minimum of the problem (1)–(2).

2 Auxiliary results

In this section we will provide several auxiliary results used in the proofs of the theorems presented in Sect. 3, namely results on attainability in integer constrained optimization problem obtained in [14].

Definition 1 We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is integer valued if $f(x) \in \mathbb{Z}$, for all $x \in \mathbb{Z}^n$.

The Theorems 1 and 2 and Corollary 1 below show that if the objective function is integer valued or, more generally, it is represented as a composition of a nondecreasing function with an integer valued function (or as a sum of functions of the latter form), then an optimal solution of the constrained integer problem is attained over any subset of integers, provided that the integer valued functions are bounded below over the feasible region.

Theorem 1 [14] *Consider the problem*

$$\min\{f_0(x) | x \in G = H \cap \mathbb{Z}^n\},$$

where H *is a subset of* $\mathbb{R}^n$ *and* f_0 *is of the form*

$$f_0(x) = h(p(x)),$$

where $p(x)$ *is an integer valued function bounded from below on* G, $(G \neq \emptyset)$ *and* h *is a nondecreasing function. Then the minimum of* f_0 *over* G *is attained at some feasible integer point.*

Examples of integer valued functions p include (quasi-)convex polynomials with integer coefficients (studied, e.g., in [1]), while the function $f_0(x) = \frac{1}{p}(\langle x, Bx\rangle)^{\frac{p}{2}}$, where $p \geq 2$, and B is positive semidefinite with $B \in \mathbb{Z}^{n\times n}$, is an example of the function of the form $f_0(x) = h(p(x))$, defined in Theorem 1.

The following corollary, where the functions f_0, h and p satisfy slightly modified assumptions follows directly from the last theorem.

Corollary 1 [14] *Consider the problem*

$$\min\{f_0(x) | x \in G = H \cap \mathbb{Z}^n\},$$

where H *is a subset of* $\mathbb{R}^n$ *and* f_0 *is of the form* $f_0(x) = h(p(x))$, *where* h *is a nondecreasing function from* $\mathbb{R}$ *onto* $(-\infty, +\infty)$ *or* $(-\infty, +\infty]$, *and* $p : \mathbb{R}^n \to \mathbb{R}$ *is an integer valued function. Then if* f_0 *is bounded from below on the nonempty feasible region* G, *the infimum of* f_0 *over* G *is attained at some feasible integer point.*

Since multivariable polynomials with integer coefficients belong to the class of integer valued function, we obtain as a special case the following corollary.

Corollary 2 *If the problem* (1)–(2)*, where* $f_i(x) = h_i(p_i(x)),\ j \in J \cup \{0\}$*,* h_i *are nondecreasing functions from* $\mathbb{R}$ *onto* $(-\infty, +\infty)$ *and* $p_i : \mathbb{R}^n \to \mathbb{R}$ *are quasi-convex polynomials with integer coefficients, is bounded below, then the objective function* f_0 *attains its infimum over the feasible region* $\mathcal{G}$*.*

Theorem 2 [14] *Consider the problem min* $\{f_0(x)|x \in G = H \cap \mathbb{Z}^n\}$*, where* H *is a subset of* $\mathbb{R}^n$ *and* f_0 *is of the form*

$$f_0(x) = \sum_{i=1}^{k} h_i(p_i(x)),$$

where h_i*,* $i = 1, 2, \ldots, k$*, are nondecreasing convex functions from* $\mathbb{R}$ *to* $(-\infty, +\infty)$*, and* $p_i(x)$*,* $i = 1, 2, \ldots, k$ *are integer valued functions bounded from below on* G*. Then the minimum of* f_0 *over* G *is attained at some feasible integer point.*

3 Generalization of the Theorem of Bell and Scarf to convex integer systems

It is well known that every inconsistent system of linear inequalities in n variables contains an inconsistent subsystem of at most $n + 1$ inequalities. The result remains true for the systems of convex inequalities, which follows directly from the Theorem of Helly [16], and it can be stated as follows: if the system

$$f_i(x) \le 0, \quad i \in J = \{1, 2, \ldots, m\} \tag{4}$$

where $f_i(x)$, $i \in J$, are convex functions, is inconsistent, then there exists an infeasible subset of constraints in (4) of cardinality not greater than $n + 1$.

Similar problem of finding an upper bound for the minimal cardinality of an infeasible subset of inconsistent integer linear system has been studied in [3] by Bell and in [18] by Scarf, who proved independently that every inconsistent system of linear inequality constraints defined over the set of integers contains an infeasible subset of constraints of cardinality not greater than 2^n.

In this section we investigate the problem of an upper bound for the minimal cardinality of the irreducible infeasible sets in the system of convex inequality constraints with integral variables. We will consider the feasible region $\mathcal{G}$ defined over the set of integers, that is the set $\mathcal{G} = \mathcal{R} \cap \mathbb{Z}^n$. Furthermore, we assume that if the functions $p_i(x)$ are convex polynomials, then all their coefficients are integer (or rational).

In the Theorem 4 we will extend the result proved by Scarf [18] and Bell [3] for linear systems to the systems of convex inequality constraints defined over the set of integer points. That is, we will show that if a convex integer program in n variables has more than 2^n linear inequality constraints, then either some of the constraints are unnecessary or there is at least one feasible integer point.

The symbol $\mathbb{IIIS}$ will be used to represent the *irreducible integer infeasible subset*, that is the subset of constraints in (2), which is infeasible in the set of integers, but for which any proper subset of constraints is feasible in $\mathbb{Z}^n$. Thus the system of constraints with indices $\mathbb{IIIS}\backslash\{i\}$, has an integer solution for any $i \in \mathbb{IIIS}$.

Definition 2 We say that the functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in J$ are $\mathcal{Q}$*-functions in the system* $f_i(x) \le 0$, $i \in J$, $x \in \mathbb{Z}^n$, if they are defined either as

1. $f_i(x) = h_i(p_i(x))$ where h_i are convex increasing functions from $\mathbb{R}$ onto $(-\infty, \infty)$ and $p_i(x)$ are convex integer valued functions or

2. f_i are of the form

$$f_i(x) = \sum_{j=1}^{k_i} h_j^i(p_j^i(x)),$$

where h_j^i, $j = 1, 2, \ldots, k_i$, are nondecreasing convex functions from $\mathbb{R}$ to $(-\infty, +\infty)$, and $p_j^i(x)$, $j = 1, 2, \ldots, k_i$ are convex integer valued functions bounded from below on the set $\{x \in \mathbb{Z}^n | f_i(x) \le 0,\ i \in J \setminus \{i\}\}$.

Indeed it follows from the Definition 2, that in case the function f_k is defined according to part (1) of the definition then the function is a $\mathcal{Q}$-function regardless of the status of the remaining constraint functions in the system. Otherwise, that is if f_k is defined according to the second part of the definition, and f_k is a $\mathcal{Q}$-function in some subsystem of the system $f_i(x) \le 0,\ i \in J$, then f_k is also a $\mathcal{Q}$-function in the latter system, although the opposite implication does not hold.

In case it is clear from the context in which system a function is considered, we will use a shorter term a "$\mathcal{Q}$-function" in the remaining part of the paper.

We will prove first the following theorem.

Theorem 3 *Suppose that $\mathbb{IIS} \subset J$ is an irreducible integer infeasible subset of the system in (2). Assume that the functions f_i, $i \in \mathbb{IIS}$ are $\mathcal{Q}$-functions in the system*

$$f_i(x) \le 0, \quad i \in \mathbb{IIS}, \quad x \in \mathbb{Z}^n, \tag{5}$$

and that the corresponding relaxed system

$$f_i(x) \le 0, \quad i \in \mathbb{IIS}, \quad x \in \mathbb{R}^n, \tag{6}$$

is feasible. Then there $\exists \epsilon_i > 0,\ \forall i \in \mathbb{IIS}$ such that the system

$$f_i(x) \le \epsilon_i, \quad x \in \mathbb{Z}^n, \quad i \in \mathbb{IIS}, \tag{7}$$

has no integer solution and the systems (7) and

$$f_i(x) < \epsilon_i, \ x \in \mathbb{Z}^n, \quad i \in \mathbb{IIS}, \tag{8}$$

are both irreducible infeasible sets.

Proof Suppose that the assumptions of the theorem are satisfied. Indeed the relaxed system

$$f_i(x) < \epsilon_i, \quad x \in \mathbb{R}^n, \quad i \in \mathbb{IIS},$$

will be feasible for any $\epsilon_i > 0$, since the system (6) has a solution. For simplicity of notation let us assume that $\mathbb{IIS} = \{1, 2, \ldots, r\}$. Let us consider the following convex integer problem

$$\min\{f_1(x) | x \in \mathbb{Z}^n, \quad f_i(x) \le 0, \quad i \in \mathbb{IIS} \backslash \{1\}\} \tag{9}$$

The feasible region of the problem defined in (9) is nonempty in the set of integers, which follows from the fact that the system with constraints in $\mathbb{IIS}$ is irreducible infeasible set. Since the system (6) has no integer solution, then the problem (9) is bounded from below (by 0). Thus by either the Corollary 1 or the Theorem 1 or 2 (depending on the form of the function $f_1(x)$), the infimum of f_1 over the feasible region defined in (9) is attained at some integer point, say x_1. Indeed $f_1(x_1) > 0$, and we define $\epsilon_1 = \frac{f_1(x_1)}{2}$. Clearly the new system

$$\begin{aligned} f_1(x) &\le \epsilon_1 \\ f_i(x) &\le 0, \quad i \in \mathbb{IIS} \backslash \{1\}, \quad x \in \mathbb{Z}^n, \end{aligned} \tag{10}$$

has no integer solution and it can be easily shown that it is still an irreducible infeasible set. Indeed, we have that for any $l \in \mathbb{IIS}\backslash\{1\}$

$$\{x \in \mathbb{Z}^n | f_i(x) \leq 0,\ i \in \mathbb{IIS}\backslash\{l\}\} \subset \{x \in \mathbb{Z}^n | f_1(x) \leq \epsilon_1, \quad f_i(x) \leq 0,\ i \in \mathbb{IIS}\backslash\{1, l\}\},$$

where the set on the left side of the inclusion is nonempty, i.e., removing any constraint in (10) would change the status of the system into a feasible one. Continuing in the above fashion, in the k-step we consider the problem

$$\begin{aligned} \min \quad & f_k(x) \\ \text{s.t. } f_i(x) & \leq \epsilon_i, \quad i = 1, 2, \ldots, k-1 \\ f_i(x) & \leq 0, \quad i = k+1, \ldots, r,\ x \in \mathbb{Z}^n, \end{aligned}$$

which has a nonempty feasible region and which has an optimal solution at some feasible integer point x_k, for which we define $\epsilon_k = \frac{f_k(x_k)}{2}$. By the construction process the system

$$\begin{aligned} f_i(x) & \leq \epsilon_i, \quad i = 1, 2, \ldots, k \\ f_i(x) & \leq 0, \quad i = k+1, \ldots, r,\ x \in \mathbb{Z}^n, \end{aligned} \tag{11}$$

is infeasible and for any $\tau \in \{1, 2, \ldots, k\}$ and $\mathcal{G}_\tau = \{x \in \mathbb{Z}^n | f_i(x) \leq 0,\ i \in \mathbb{IIS}\backslash\{\tau\}\}$ we have

$$\mathcal{G}_\tau \subset \{x \in \mathbb{Z}^n | f_i(x) \leq \epsilon_i, \quad i \in \{1, 2, \ldots, k\}\backslash\{\tau\}, \quad f_i(x) \leq 0, i = k+1, \ldots, r\},$$

and for $\tau \in \{k+1, \ldots, r\}$

$$\mathcal{G}_\tau \subset \{x \in \mathbb{Z}^n | f_i(x) \leq \epsilon_i, \quad i = 1, 2, \ldots, k,\ f_i(x) \leq 0, \quad i \in \{k+1, \ldots, r\} \setminus \{\tau\}\}.$$

Since the system (5) is an irreducible integer infeasible set, then $\mathcal{G}_\tau \neq \emptyset$, which implies that removing any constraint in the system (11) transforms this system into the feasible one, i.e., for all $k \in \mathbb{IIS}$ the system (11) is an irreducible integer infeasible set. Thus, after r steps we obtain in particular that the system (7) is an irreducible infeasible set.

To show that the system (8) is an irreducible integer infeasible set, we observe that for any $l \in \mathbb{IIS}$ the following inclusion holds for $A_l = \{x \in \mathbb{Z}^n | f_i(x) \leq 0,\ i \in \mathbb{IIS}\backslash\{l\},\ f_l(x) > 0\}$

$$A_l \subset \{x \in \mathbb{Z}^n | f_i(x) < \epsilon_i, \quad i \in \mathbb{IIS}\backslash\{l\}, \quad f_l(x) > 0\}.$$

But $A_l \neq \emptyset$, since the system $f_i(x) \leq 0,\ i \in \mathbb{IIS},\ x \in \mathbb{Z}^n$ is an irreducible integer infeasible set. Therefore, the set on the right side of the latter inclusion, i.e., the set

$$B_l = \{x \in \mathbb{Z}^n | f_i(x) < \epsilon_i, \quad i \in \mathbb{IIS}\backslash\{l\}, \quad f_l(x) > 0\}$$

is nonempty as well. Furthermore, it follows from the construction process of $\epsilon_i,\ i \in \mathbb{IIS}$, that there are no integer points in the region (7), which implies that the set

$$\{x \in \mathbb{Z}^n | f_i(x) < \epsilon_i, \quad i \in \mathbb{IIS}\backslash\{l\}, \quad 0 < f_l(x) \leq \epsilon_l\},$$

is also empty. Thus, we have that $B_l \neq \emptyset,\ l \in \mathbb{IIS}$, which implies that removing lth constraint from the system (8) changes the status of the system to the feasible one. This completes the proof of the theorem. □

Let us now consider the case when the system (6) is not necessarily irreducible integer infeasible system. We will show that the following corollary follows from the Theorem 3.

Corollary 3 *Assume that the functions* $f_i(x),\ i \in J$, *are* $\mathcal{Q}$*-functions and that the system*

$$f_i(x) \le 0, \quad i \in J, \quad x \in \mathbb{R}^n,$$

has a real solution but it does not have an integer solution. Then there $\exists \epsilon_i \ge 0,\ \forall i \in J$, *where* $\epsilon = (\epsilon_1, \dots, \epsilon_m) \ne 0$ *such that the system*

$$f_i(x) \le \epsilon_i, \quad i \in J,$$

has no integer solution.

Proof Let $\mathbb{IIS}$ be any irreducible integer infeasible subset of the system with the constraints with indices in J. Then applying Theorem 3 to this subsystem we obtain that $\exists \epsilon_i > 0,\ i \in \mathbb{IIS}$, such that the system

$$\begin{aligned} f_i(x) &\le \epsilon_i, \quad x \in \mathbb{R}^n, \quad i \in \mathbb{IIS}, \\ f_i(x) &\le 0, \quad i \in J \backslash \mathbb{IIS}, \end{aligned}$$

has no integer solution. □

Theorem 4 *Assume that the functions* $f_i,\ i \in J$ *are* $\mathcal{Q}$*-functions. If the system*

$$f_i(x) \le 0, \quad i \in J, \ x \in \mathbb{Z}^n, \tag{12}$$

is inconsistent, then there exists an infeasible subset of constraints in (12) of cardinality not greater than 2^n, *that is*

$$\min |\mathbb{IIS}| \le 2^n.$$

Proof To prove the theorem we will show that any irreducible integer infeasible subset of constraints in (12) (denoted by $\mathbb{IIS}$), contains at most 2^n constraints. We assume that the system $f_i(x) \le 0,\ i \in \mathbb{IIS},\ x \in \mathbb{R}^n$, is feasible, while the system $f_i(x) \le 0,\ i \in \mathbb{IIS},\ x \in \mathbb{Z}^n$, has no integer solution. We note that such an assumption is justified, since in case the corresponding relaxed system has no real solution, the converse of the Theorem of Helly yields that $\min |\mathbb{IIS}| \le n + 1$, which given that $n + 1 \le 2^n,\ \forall n \in \mathbb{N}$, would complete the proof of the theorem. Suppose that $\epsilon_i,\ i \in \mathbb{IIS}$ are obtained by the construction process described in the proof of the Theorem 3. Since $\epsilon_i > 0,\ i \in \mathbb{IIS}$, then the system $f_i(x) < \epsilon_i,\ i \in \mathbb{IIS},\ x \in \mathbb{R}^n$ has a solution. Let $\bar{x}_i \in \mathbb{Z}^n,\ i \in \mathbb{IIS}$ be such that

$$f_i(\bar{x}_i) = \min \{ f_i(x) | f_i(x) \ge \epsilon_i, \ f_j(x) < \epsilon_j, \ j \in \mathbb{IIS} \backslash \{i\} \}. \tag{13}$$

The feasible region of the problem (13) is nonempty, and the problem is bounded below. Therefore, by either Theorem 1, 2 or Corollary 1 or 2 (depending on the form of the function $f_i(x)$), the function $f_i(x)$ attains its infimum at some feasible integer point, say $\bar{x}_i$. Indeed all points $\bar{x}_i,\ i \in \mathbb{IIS}$ are distinct, since each of them satisfies

$$\begin{aligned} f_i(\bar{x}_i) &\ge \epsilon_i, \\ f_j(\bar{x}_i) &< \epsilon_j, \quad j \in \mathbb{IIS} \backslash \{i\}. \end{aligned}$$

Suppose for simplicity of notation that $\mathbb{IIS} = \{1, 2, \dots, r\}$. Let us consider the convex hull of the integer points $\bar{x}_i,\ i = 1, 2, \dots, r$, denoted as $H_0 = \text{conv}\{\bar{x}_i,\ i = 1, 2, \dots, r\}$, and let us define the set $\hat{G} = \{x \in \mathbb{R}^n | f_i(x) < f_i(\bar{x}_i),\ i \in \mathbb{IIS}\}$. It follows from the fact that each $\bar{x}_i$ is a solution to the problem (13) and convexity of $\hat{G}$, that $H_0 \setminus \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_r\} \subset \hat{G}$. Furthermore, since the set $\hat{G}$ has no integer points, the same is true about the set $H_0 \setminus \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_r\}$. If $r > 2^n$, then given that all points $\bar{x}_i$ are distinct, for at least two points

$\bar{x}_\kappa, \bar{x}_\tau, (\bar{x}_\kappa \neq \bar{x}_\tau)$ in the set $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r\}$ we have $\bar{x}_\kappa \equiv \bar{x}_\tau$ (modulo 2), (i.e., the coordinates of $\bar{x}_\kappa$ and $\bar{x}_\tau$ differ only by an even number). This shows that $\tilde{x} = \frac{\bar{x}_\kappa - \bar{x}_\tau}{2}$ is an integer point. Thus $\tilde{x} + \bar{x}_\tau = \frac{\bar{x}_\kappa - \bar{x}_\tau}{2} + \bar{x}_\tau = \frac{\bar{x}_\kappa + \bar{x}_\tau}{2}$, which indicates that the midpoint of the line segment joining $\bar{x}_\kappa$ and $\bar{x}_\tau$ is an integer point. On the other hand $\frac{\bar{x}_\kappa + \bar{x}_\tau}{2} \in H_0 \backslash \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r\} \subset \hat{G}$, a contradiction, since $\hat{G}$ contains no integer points. □

Corollary 4 *If there are $2^n + q$, (where $q \geq 1$), constraints in the system defining the region*

$$\mathcal{G} = \{x \in \mathbb{Z}^n | f_i(x) \leq 0, \ i \in J\},$$

(where the functions f_i, $i \in J$ are $\mathcal{Q}$-functions), then either at least q constraints in the system are redundant, or the set $\mathcal{G}$ contains at least one integer point.

Proof Follows from the Theorem 4. □

The following theorem is a generalization of the result proved in Theorem 3 (more specifically of its converse).

Theorem 5 *If there are more than κ^n, (where $\kappa \geq 2$, $\kappa \in \mathbb{N}$) necessary constraints in the system defining the set*

$$\mathcal{G} = \{x \in \mathbb{Z}^n | f_i(x) \leq 0, \ i \in J\},$$

(where the functions f_i are $\mathcal{Q}$-functions), then the set $\mathcal{G}$ contains at least $\kappa - 1$ integer points located on the same straight-line.

Proof Without a loss of generality we can assume that all constraints with indices in J are necessary, since otherwise the redundant constraints could be removed without affecting the feasible region.

If $\kappa = 2$ the proof follows directly from Theorem 4. Suppose now that $\kappa = 3$. Since $3^n > 2^n$ then by Corollary 4 (using $q = 1$), the set $\mathcal{G}$ contains at least one integer point. If the region $\mathcal{G}$ contains two or more integer points, the Theorem 5 holds. Thus, we can suppose that the region $\mathcal{G}$ contains exactly one integer point, say x_0. Then applying the argument outlined in the proof of Theorem 3 allows us to conclude that there exist numbers $\epsilon_i > 0$, $i \in J$, such that the region

$$f_i(x) < \epsilon_i, \quad x \in \mathbb{Z}^n, \quad i \in J,$$

does not include any other integer point than x_0. Furthermore, process similar to the one presented in the proof of the Theorem 4 allows us to obtain m distinct integer points $\bar{x}_i$, $i = 1, 2, \ldots, m$, (where $\bar{x}_i \neq x_0$, $i = 1, 2, \ldots, m$), whose convex hull (except for the points $\bar{x}_i$, $i = 1, 2, \ldots, m$) is contained in the open convex region $\hat{G} = \{x \in \mathbb{R}^n | f_i(x) < f_i(\bar{x}_i), \ i \in J\}$, containing no other integer points than x_0.

Since there are 3^n distinct vectors in $\mathbb{Z}^n$ with coordinates 0, 1, or 2, and $m > 3^n$, then there exist i, j, such that $\bar{x}_i \equiv \bar{x}_j$ (modulo 3). This means that the vectors $\frac{2}{3}\bar{x}_i + \frac{1}{3}\bar{x}_j$ and $\frac{1}{3}\bar{x}_i + \frac{2}{3}\bar{x}_j$ are integer points belonging to the open set $\hat{G}$, a contradiction, since $\hat{G}$ contains only one integer point.

In general, if the number of necessary constraints m is greater than κ^n, i.e., $m > \kappa^n$, assuming that the region $\mathcal{G}$ contains no more than $\kappa - 2$ integer points and using argument similar to the one made above, will allow us to deduce that there exist i, j, such that $\bar{x}_i \equiv \bar{x}_j$ (modulo κ). This will imply that the following $\kappa - 1$ vectors

$$\frac{1}{\kappa}\bar{x}_i + \frac{\kappa - 1}{\kappa}\bar{x}_j, \frac{2}{\kappa}\bar{x}_i + \frac{\kappa - 2}{\kappa}\bar{x}_j, \ldots, \frac{\kappa - 1}{\kappa}\bar{x}_i + \frac{1}{\kappa}\bar{x}_j,$$

are integer points lying on the same straight-line (namely in the interior of the line segment joining the points $\bar{x}_i$ and $\bar{x}_j$), and that all are contained in the interior of the region $\hat{G}$, which by assumption contains no more than $\kappa - 2$ integer points, a contradiction. □

Theorem 6 *Suppose that the functions f_i, $i \in J$, are Q-functions and that the problem*

$$\min\{f_0(x) | f_i(x) \le 0,\ i \in J,\ x \in \mathbb{Z}^n\},$$

is bounded below. Then

$$\min\{f_0(x) | f_i(x) \le 0,\ i \in J,\ x \in \mathbb{Z}^n\} = \min\{f_0(x) | f_i(x) \le 0,\ i \in I,\ x \in \mathbb{Z}^n\} \quad (14)$$

for some subsystem $f_i(x) \le 0$, $i \in I$ of the system

$$f_i(x) \le 0, \quad i \in J, \quad (15)$$

with no more than $2^n - 1$ inequality constraints, i.e., with $|I| \le 2^n - 1$, and the minimum in both problems in (14) are attained at some feasible integer point(s).

Proof Let $\tilde{f}_0 = \min\{f_0(x) | f_i(x) \le 0,\ i \in J,\ x \in \mathbb{Z}^n\}$. Then $\forall j \in \mathbb{N}$, the system

$$f_i(x) \le 0, \quad i \in J, \quad f_0(x) \le \tilde{f}_0 - \frac{1}{j} \quad (16)$$

has no integer solution. Therefore by Theorem 4, for all $j \in \mathbb{N}$ there is a subsystem of the system (16) with at most 2^n constraints having no integer solution. Since the system (15) has an integer solution, each such subsystem contains the constraint $f_0(x) \le \tilde{f}_0 - \frac{1}{j}$. Hence there is a subset $I \subset J$, satisfying $|I| \le 2^n - 1$ such that the system $f_i(x) \le 0,\ i \in I,\ f_0(x) \le \tilde{f}_0 - \frac{1}{j}$ has no integer solution for all $j \in \mathbb{N}$. Consequently $f_i(x) \le 0,\ i \in I,\ f_0(x) < \tilde{f}_0$ has no integer solution, which along with the fact that

$$\min\{f_0(x) | f_i(x) \le 0,\ i \in J,\ x \in \mathbb{Z}^n\} \ge \min\{f_0(x) | f_i(x) \le 0,\ i \in I,\ x \in \mathbb{Z}^n\}$$

(since $I \subset J$), implies the equality (14). Attainability of the minima in (14) follows from either the Theorem 1, or Theorem 2 or from the Corollary 1, depending on the form of the function f_0. □

We remark that the issue of boundedness of the problem (14) (i.e., of the problem (1)–(2) was considered in [14], where we have shown in particular that in case the functions f_i, $i \in J$ are either faithfully convex (satisfying some mild assumption) or quasi-convex polynomials, the integer problem (1)–(2) is bounded below if and only if the corresponding relaxed (continuous) problem is bounded below.

The following corollary follows from the Corollary 3.

Corollary 5 *Let us assume that $f_i(x),\ i \in J$, are $\mathcal{Q}$-functions, and let us define*

$$\mathcal{G}(\epsilon) = \{x \in \mathbb{Z}^n | f_i(x) \le \epsilon_i,\ i \in J\}.$$

for $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$. Suppose that for arbitrary sequence $\{\epsilon^k\}$, ($\epsilon^k \neq 0$) of nonnegative vectors ϵ^k approaching zero, the sets $\mathcal{G}(\epsilon^k)$ are nonempty, i.e., each of them contains an integer point. Then the set $\mathcal{G}$, (defined in (2)) is also nonempty in the set of integers.

Remark Corollary 5 similarly to the Theorem 3 can not be generalized to convex functions of the form $f(x) = h(p(x))$ where h and g are not satisfying assumptions stated in the definition of a $\mathcal{Q}$-function, i.e., where h is neither onto $\mathbb{R}$ nor g is bounded below over the set of remaining constraints. For example the set $\mathcal{G}_0(\epsilon_k) = \{x \in \mathbb{Z} | e^{-x} \le \epsilon_k\}$, is feasible for any $\epsilon_k > 0$, but is infeasible for $\epsilon_k = 0$. Similarly, the function $f(x) = e^{-x}$ has an unattained unconstrained infimum in the set of integers, equal 0. Indeed, the function $f(x) = h(p(x))$ does not belong to the class of convex functions considered in this paper, since $h(x) = e^x$ is not onto $\mathbb{R}$, and $p(x) = -x$ is not bounded below on the set of integers.

We note that result analogous to Corollary 5 was earlier stated in [13] for systems of faithfully convex and/or quasi-convex inequality constraints in case the variable x is continuous.

Acknowledgments The author would like to thank two anonymous referees whose helpful comments and suggestions led to an improved presentation of this paper.

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